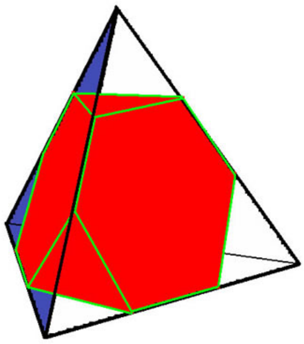
CCA Math Bonanza

Individual Round

1. Michael the Mouse finds a block of cheese in the shape of a regular tetrahedron (a pyramid with equilateral triangles for all faces). He cuts some cheese off each corner with a sharp knife. How many faces does the resulting solid have?

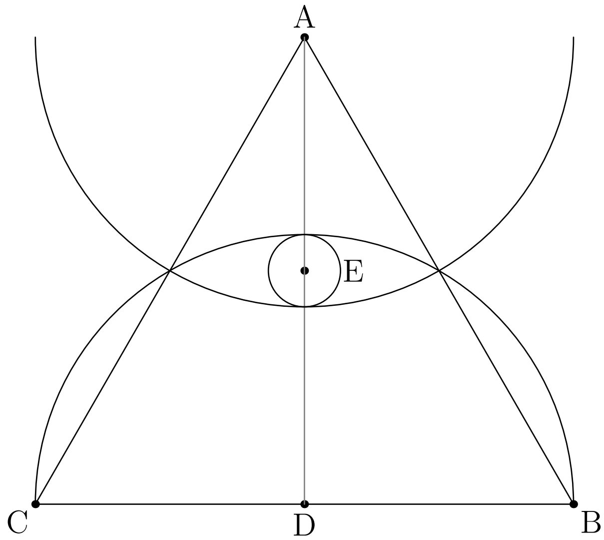


2. The operation $*$ is defined by the following: $a * b = a! - ab - b$. Compute the value of $5 * 8$.

3. Mark's teacher is randomly pairing his class of 16 students into groups of 2 for a project. What is the probability that Mark is paired up with his best friend, Mike? (There is only one Mike in the class)

4. Kevin the Koala eats 1 leaf on the first day of his life, 3 leaves on the second, 5 on the third, and in general eats $(2n-1)$ leaves on the nth day. What is the smallest positive integer $n > 1$ such that the total number of leaves Kevin has eaten his entire n-day life is a perfect sixth power?

5. Triangle ABC is equilateral with side length 12. Point D is the midpoint of side $\overline{BC}$. Circles A and D intersect at the midpoints of sides AB and AC. Point E lies on segment $\overline{AD}$ and circle E is internally tangent to circles A and D. Compute the radius of circle E.



6. How many positive integers less than or equal to 1000 are divisible by 2 and 3 but not by 5?

7. Harry Potter would like to purchase a new owl which cost him 2 Galleons, a Sickle, and 5 Knuts. There are 23 Knuts in a Sickle and 17 Sickles in a Galleon. He currently has no money, but has many potions, each of which are worth 9 Knuts. How many potions does he have to exhange to buy this new owl?

8. A rectangle has an area of 16 and a perimeter of 18; determine the length of the diagonal of the rectangle.

9. There is 1 integer in between 300 and 400 (base 10) inclusive such that its last digit is 7 when written in bases 8, 10, and 12. Find this integer, in base 10.

10. The fourth-degree equation $x^4 - x - 504 = 0$ has 4 roots r_1, r_2, r_3, r_4. If S_x denotes the value of $r_1^x + r_2^x + r_3^x + r_4^x$, compute S_4.

11. A dog owns 4 different color shoes and 4 identical green socks. He can fit every shoe and sock on each of his four distinguishable paws. In how many different orders can he put on the shoes and socks, provided that on each paw he must put on the sock before the shoe?

12. Positive integers x, y, z satisfy $x^3 + xy + x^2 + xz + y + z = 301$. Compute $y + z - x$.

13. Let $ABCD$ be a tetrahedron such that $AD \perp BD$, $BD \perp CD$, $CD \perp AD$ and $AD = 10$, $BD = 15$, $CD = 20$. Let E and F be points such that E lies on BC, $DE \perp BC$, and $ADEF$ is a rectangle. If S is the solid consisting of points inside $ABCD$ but outside $FBCD$, compute the volume of S.

14. 10 children each have a lunchbox which they store in a basket before entering their classroom. However, being messy children, their lunchboxes get mixed up. When leaving the classroom each student picks up a lunchbox at random. Define a *cyclic triple* of students (A, B, C) to be three distinct students such that A has B's lunchbox, B has C's lunchbox, and C has A's lunchbox. Two cyclic triples are considered the same if they contain the same three students (even if in a different order). Determine the expected value of the number of cyclic triples.

15. Let ω_1 and ω_2 be circles with radii 3 and 12 and externally tangent at point P. Let a common external tangent intersect ω_1, ω_2 at S, T respectively and the common internal tangent at point Q. Define X to be the point on $\overrightarrow{QP}$ such that $XQ = 10$. If XS, XT intersect ω_1, ω_2 a second time at A and B, determine $\tan \angle APB$