2017 Math Bonanza Individual Round

CCA Math Team

January 14, 2017

Time limit: 75 minutes

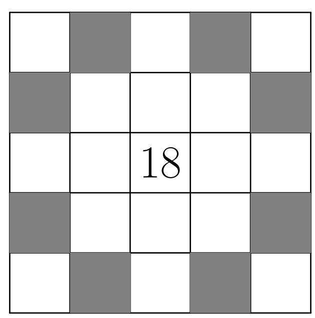
I1) Find the integer n such that $6! \times 7! = n!$.

I2) A rectangle is inscribed in a circle of area 32π and the area of the rectangle is 34. Find its perimeter.

I3) A sequence starts with 2017 as its first term and each subsequent term is the sum of cubes of the digits in the previous number. What is the 2017th term of this sequence?

I4) Cole is trying to solve the *Collatz conjecture*. She decides to make a model with a piece of wood with a hole for every natural number. For every even number there is a rope from n to $\frac{n}{2}$ and for every odd number there is a rope from n to $3n+1$. She wants to bring her model to a convention but in order to do that she needs to cut off the part containing the first 240 holes. How many ropes did she break?

I5) In the *magic square* below, every integer from 1 to 25 can be filled in such that the sum in every row, column, and long diagonal is the same. Given that the number in the center square is 18, what is the sum of the entries in the shaded squares?



I6) Determine the largest prime which divides both $2^{24}-1$ and $2^{16}-1$.

I7) Ari the Archer is shooting at an abnormal target. The target consists of 100 concentric rings, each of width 1, so that the total radius of the target is 100. The point value of a given ring of the target is equal to its area (so getting a bull's eye would be worth π points, but hitting on the outer ring would give 199π points). Given that Ari hits any point on the target uniformly at random, what is his expected score?

I8) Let $a_1, a_2, \ldots, a_{18}$ be a list of prime numbers such that $\frac{1}{64} \times a_1 \times a_2 \times \cdots \times a_{18}$ is one million. Determine the sum of all positive integers n such that

$$\sum_{i=1}^{18} \frac{1}{\log_{a_i} n}$$

is a positive integer.

I9) Magic Mark performs a magic trick using a standard 52-card deck except the suits are erased from cards (so that there are 4 identical cards of each rank). He randomly takes 13 cards and uses those to perform his trick. He lets you randomly pick a card from those 13, memorize it, and put it back in the pile of 13 cards. He then shuffles the 13 and takes out a card randomly. If he picks a card identical to yours, the trick is successful. What is probability that the trick is successful?

I10) Find the sum of the two smallest possible values of x° (in degrees) that satisfy the following equation if x is greater than 2017°:

$$\cos^5 9x + \cos^5 x = 32\cos^5 5x \cos^5 4x + 5\cos^2 9x \cos^2 x(\cos 9x + \cos x)$$

I11) 480 1 cm unit cubes are used to build a block measuring 6 cm by 8 cm by 10 cm. A tiny ant then chews his way in a straight line from one vertex of the block to the furthest vertex. How many cubes does the ant pass through? The ant is so tiny that he does not "pass through" cubes if he is merely passing through where their edges or vertices meet.

I12) Let $a_1, a_2, \ldots, a_{2017}$ be the 2017 distinct complex numbers which satisfy $a_i^{2017} = a_i + 1$ for $i = 1, 2, \ldots, 2017$. Compute

$$\sum_{i=1}^{2017} \frac{a_i}{a_i^2 + 1}.$$

I13) Toner Drum and Celery Hilton are both running for president. A total of 129 million people cast their vote in a random order, with exactly 63 million and 66 million voting for Toner Drum and Celery Hilton, respectively. The Combinatorial News Network displays the face of the leading candidate on the front page of their website. If the two candidates are tied, both faces are displayed. What is the probability that Toner Drum's face is never displayed on the front page?

I14) Find a pair (x, y) of positive integers $x < y$ such that

$$37^2 + 46^2 + 49^2 - 20^2 - 17^2 = x^2 + y^2.$$

Note: there may be several answers; just provide one of them.

I15) Let ABC, $AB < AC$ be an acute triangle inscribed in circle Γ with center O. The altitude from A to BC intersects Γ again at A_1. OA_1 intersects BC at A_2 Similarly define B_1, B_2, C_1, and C_2. Then $B_2C_2 = 2\sqrt{2}$. If B_2C_2 intersects AA_2 at X and BC at Y, then $XB_2 = 2$ and $YB_2 = k$. Find k^2.