CCA Math Bonanza Lightning Round

CCA Math Team

January 14, 2017

Round 1

L1.1) Consider the harmonic sequence $\frac{2017}{4}, \frac{2017}{7}, \frac{2017}{10}, \ldots$, where the reciprocals of the terms of the sequence form an arithmetic sequence. How many terms of this sequence are integers?

L1.2) How many ways are there to rearrange the letters of CCARAVEN?

L1.3) Triangle ABC has points A at $(0,0)$, B at $(9,12)$, and C at $(-6,8)$ in the coordinate plane. Find the length of the angle bisector of $\angle BAC$ from A to where it intersects BC.

L1.4) Wild Bill goes to Las Vejas and takes part in a special lottery called *Reverse Yrettol*. In this lottery, a player may buy a ticket on which he or she may select 5 distinct numbers from 1-20 (inclusive). Then, 5 distinct numbers from 1-20 are drawn at random. A player wins if his or her ticket contains *none* of the numbers which were drawn. If Wild Bill buys a ticket, what is the probability that he will win?

Round 2

L2.1) Adam and Mada are playing a game of one-on-one basketball, in which participants may take 2-point shots (worth 2 points) or 3-point shots (worth 3 points). Adam makes 10 shots of either value while Mada makes 11 shots of either value. Furthermore, Adam made the same number of 2-point shots as Mada made 3-point shots. At the end of the game, the two basketball players realize that they have the exact same number of points! How many total points were scored in the game?

L2.2) Non-degenerate triangle ABC has $AB = 20$, $AC = 17$, and $BC = n$, an integer. How many possible values of n are there?

L2.3) Jack is jumping on the number line. He first jumps one unit and every jump after that he jumps one unit more than the previous jump. What is the least amount of jumps it takes to reach exactly 19999 from his starting place?

L2.4) Define $f(n) = \text{LCM}(1, 2, \ldots, n)$. Determine the smallest positive integer a such that $f(a) = f(a+2)$.

Round 3

L3.1) Express $2.3\overline{57}$ as a common fraction.

L3.2) Bob is flipping bottles. Each time he flips the bottle, he has a 0.25 probability of landing it. After successfully flipping a bottle, he has a 0.8 probability of landing his next flip. What is the expected value of the number of times he has to flip the bottle in order to flip it twice in a row?

L3.3) An acute triangle ABC has side lenghths a, b, c such that a, b, c forms an arithmetic sequence. Given that the area of triangle ABC is an integer, what is the smallest value of its perimeter?

L3.4) A random walk is a process in which something moves from point to point, and where the direction of movement at each step is randomly chosen. Suppose that a person conducts a random walk on a line: he starts at 0 and each minute randomly moves either 1 unit in the positive direction or 1 unit in the negative direction. What is his expected distance from the origin after 6 moves?

Round 4

L4.1) Compute

$$\sum_{k=0}^{\infty} k \left(\frac{1}{3}\right)^k .$$

L4.2) Find $\arctan(1) + \arctan(2) + \arctan(3)$ in radians.

L4.3) Let $f(x)$ be the greatest prime number at most x. Let $g(x)$ be the least prime number greater than x. Find

$$\sum_{i=2}^{100} \frac{1}{f(i)\, g(i)}.$$

L4.4) Let ABC be an acute triangle. $PQRS$ is a rectangle with P on AB, Q and R on BC, and S on AC such that $PQRS$ has the largest area among all rectangles $TUVW$ with T on AB, U and V on BC, and W on AC. If D is the point on BC such that $AD \perp BC$, then PQ is the harmonic mean of $\frac{AD}{DB}$ and $\frac{AD}{DC}$. What is BC?

Note: The harmonic mean of two numbers a and b is the reciprocal of the arithmetic mean of the reciprocals of a and b.

Round 5

L5.1) Find $x + y + z$ when

$$a_1x + a_2y + a_3z = a$$
$$b_1x + b_2y + b_3z = b$$
$$c_1x + c_2y + c_3z = c$$

Given that

$$a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1) = 9$$
$$a(b_2c_3 - b_3c_2) - a_2(bc_3 - b_3c) + a_3(bc_2 - b_2c) = 17$$
$$a_1(bc_3 - b_3c) - a(b_1c_3 - b_3c_1) + a_3(b_1c - bc_1) = -8$$
$$a_1(b_2c - bc_2) - a_2(b_1c - bc_1) + a(b_1c_2 - b_2c_1) = 7.$$

L5.2) Compute $e^\pi + \pi^e$. If your answer is A and the correct answer is C, then your score on this problem will be $\frac{4}{\pi}\arctan\left(\frac{1}{|C-A|}\right)$ (note that the closer you are to the right answer, the higher your score is).

L5.3) How many ways are there to fill a $3 \times 3 \times 6$ rectangular prism with $1 \times 1 \times 2$ blocks? Rotations are not distinct. If your answer is A and the correct answer is C, then your score on this problem will be $\max\left(2\left(1 - \left|\frac{C-A}{C}\right|\right), 0\right)$.

L5.4) In the game of Colonel Blotto, you have 100 troops to distribute among 10 castles. Submit a 10-tuple $(x_1, x_2, \ldots x_{10})$ of nonnegative integers such that $x_1 + x_2 + \ldots + x_{10} = 100$, where each x_i represent the number of troops you want to send to castle i. Your troop distribution will be matched up against each opponent's and you will win 10 points for each castle that you send more troops to (if you send the same number, you get 5 points, and if you send fewer, you get none). Your aim is to score the most points possible averaged over all opponents.

For example, if team A submits $(90, 10, 0, \ldots, 0)$, team B submits $(11, 11, 11, 11, 11, 11, 11, 11, 11, 1)$, and team C submits $(10, 10, 10, \ldots 10)$, then team A will win 10 points against team B and 15 points against team C, while team B wins 90 points against team C. Team A averages 12.5 points, team B averages 90 points, and team C averages 47.5 points.