2016 Math Bonanza Individual Round

CCA Math Team

May 28, 2016

Time Limit: 75 minutes

I1) Compute the integer
$$\frac{2^{\left(2^5-2\right)/5-1}-2}{5}.$$

I2) Rectangle $ABCD$ has perimeter 178 and area 1848. What is the length of the diagonal of the rectangle?

I3) Amanda has the list of even numbers $2, 4, 6, \ldots 100$ and Billy has the list of odd numbers $1, 3, 5, \ldots 99$. Carlos creates a list by adding the square of each number in Amanda's list to the square of the corresponding number in Billy's list. Daisy creates a list by taking twice the product of corresponding numbers in Amanda's list and Billy's list. What is the positive difference between the sum of the numbers in Carlos's list and the sum of the numbers in Daisy's list?

I4) The three digit number $n = CCA$ (in base 10), where $C \neq A$, is divisible by 14. How many possible values for n are there?

I5) Let ABC be a triangle with $AB = 3$, $BC = 4$, and $AC = 5$. If D is the projection from B onto AC, E is the projection from D onto BC, and F is the projection from E onto AC, compute the length of the segment DF.

I6) Let a, b, c be non-zero real numbers. The lines $ax + by = c$ and $bx + cy = a$ are perpendicular and intersect at a point P such that P also lies on the line $y = 2x$. Compute the coordinates of point P.

I7) Simon is playing chess. He wins with probability $1/4$, loses with probability $1/4$, and draws with probability $1/2$. What is the probability that, after Simon has played 5 games, he has won strictly more games than he has lost?

I8) Let $f(x) = x^2 + x + 1$. Determine the ordered pair (p, q) of primes satisfying $f(p) = f(q) + 242$.

I9) Let $P(X) = X^5 + 3X^4 - 4X^3 - X^2 - 3X + 4$. Determine the number of monic polynomials $Q(x)$ with integer coefficients such that $\frac{P(X)}{Q(X)}$ is a polynomial with integer coefficients. Note: a monic polynomial is one with leading coefficient 1 (so $x^3 - 4x + 5$ is one but not $5x^3 - 4x^2 + 1$ or $x^2 + 3x^3$).

I10) Let ABC be a triangle with $AC = 28$, $BC = 33$, and $\angle ABC = 2\angle ACB$. Compute the length of side AB.

I11) How many ways are there to place 8 1s and 8 0s in a 4×4 array such that the sum in every row and column is 2?

1	0	0	1
0	1	1	0
0	1	1	0
1	0	0	1

I12) Let f be a function from the set $X = \{1, 2, \ldots, 10\}$ to itself. Call a partition (S, T) of X $f-balanced$ if for all $s \in S$ we have $f(s) \in T$ and for all $t \in T$ we have $f(t) \in S$. (A partition (S, T) is a pair of subsets S and T of X such that $S \cap T = \emptyset$ and $S \cup T = X$. Note that (S, T) and (T, S) are considered the same partition).

Let $g(f)$ be the number of $f-balanced$ partitions, and let m equal the maximum value of g over all functions f from X to itself. If there are k functions satisfying $g(f) = m$, determine $m + k$.

I13) Let $P(x)$ be a polynomial with integer coefficients, leading coefficient 1, and $P(0) = 3$. If the polynomial $P(x)^2 + 1$ can be factored as a product of two non-constant polynomials with integer coefficients, and the degree of P is as small as possible, compute the largest possible value of $P(10)$.

I14) Compute

$$\sum_{k=1}^{420} \gcd(k, 420).$$

I15) Let ABC be a triangle with $AB = 5$, $AC = 12$ and incenter I. Let P be the intersection of AI and BC. Define ω_B and ω_C to be the circumcircles of ABP and ACP, respectively, with centers O_B and O_C. If the reflection of BC over AI intersects ω_B and ω_C at X and Y, respectively, then $\frac{O_B O_C}{XY} = \frac{PI}{IA}$. Compute BC.