2016 Math Bonanza Lightning Round

1 Set 1

L1.1) What is the sum of all the integers n such that $|n-1| < \pi$?

L1.2) What is the largest prime factor of $729 - 64$?

L1.3) If the GCD of a and b is 12 and the LCM of a and b is 168, what is the value of $a \times b$?

L1.4) A triangle has a perimeter of 4 *yards* and an area of 6 square $feet$. If one of the angles of the triangle is right, what is the length of the largest side of the triangle, in feet?

2 Set 2

L2.1) Bhairav runs a 15-mile race at 28 miles per hour, while Daniel runs at 15 miles per hour and Tristan runs at 10 miles per hour. What is the greatest length of time, in *minutes*, between consecutive runners' finishes?

L2.2) In triangle ABC, $AB = 7$, $AC = 9$, and $BC = 8$. The angle bisector of $\angle BAC$ intersects side BC at D, and the angle bisector of $\angle ABC$ intersects AD at E. Compute AE^2.

L2.3) Let ABC be a right triangle with $\angle ACB = 90^\circ$. D is a point on AB such that $CD \perp AB$. If the area of triangle ABC is 84, what is the smallest possible value of

$$AC^2 + (3 \cdot CD)^2 + BC^2?$$

L2.4) What is the largest integer that must divide $n^5 - 5n^3 + 4n$ for all integers n?

3 Set 3

L3.1) How many 3-digit positive integers have the property that the sum of their digits is greater than the product of their digits?

L3.2) Let $a_0 = 1$ and define the sequence $\{a_n\}$ by

$$a_{n+1} = \frac{\sqrt{3}a_n - 1}{a_n + \sqrt{3}}.$$

If a_{2017} can be expressed in the form $a + b\sqrt{c}$ in simplest radical form, compute $a + b + c$.

L3.3) Triangle ABC has side length $AB = 5$, $BC = 12$, and $CA = 13$. Circle Γ is centered at point X exterior to triangle ABC and passes through points A and C. Let D be the second intersection of Γ with segment $\overline{BC}$. If $\angle BDA = \angle CAB$, the radius of Γ can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m + n$.

L3.4) Let S be the set of the reciprocals of the first 2016 positive integers and T the set of all subsets of S that form arithmetic progressions. What is the largest possible number of terms in a member of T?

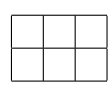
4 Set 4

L4.1) Determine the remainder when

$$2^6 \cdot 3^{10} \cdot 5^{12} - 75^4 \left(26^2 - 1\right)^2 + 3^{10} - 50^6 + 5^{12}$$

is divided by 1001.

L4.2) Consider the 2×3 rectangle below. We fill in the small squares with the numbers $1, 2, 3, 4, 5, 6$ (one per square). Define a *tasty* filling to be one such that each row is **not** in numerical order from left to right and each column is **not** in numerical order from top to bottom. If the probability that a randomly selected filling is tasty is $\frac{m}{n}$ for relatively prime positive integers m and n, what is $m + n$?



L4.3) Let ABC be a non-degenerate triangle with perimeter 4 such that $a = bc\sin^2 A$. If M is the maximum possible area of ABC and m is the minimum possible area of ABC, then $M^2 + m^2$ can be expressed in the form $\frac{a}{b}$ for relatively prime positive integers a and b. Compute $a + b$.

L4.4) Real numbers $X_1, X_2, \ldots, X_{10}$ are chosen uniformly at random from the interval [0,1]. If the expected value of $\min(X_1, X_2, \ldots, X_{10})^4$ can be expressed as a rational number $\frac{m}{n}$ for relatively prime positive integers m and n, what is $m + n$?

Question) Eshaan the Elephant has a long memory. He remembers that out of the integers $0, 1, 2, \ldots, 15$, one of them is special. Submit to the grader an ordered 4-tuple of subsets of $0, 1, 2, \ldots, 15$ and they will tell you whether the special number is in each. You can then submit your guess for the special number on the next round for points. (You might want to write down a copy of your submission somewhere other than your answer sheet. Note that this question itself is not worth any points, though the corresponding problem in Set 5 is.)

5 Set 5

L5.1) Eshaan the Elephant has a long memory. He remembers that out of the integers $0, 1, 2, \ldots, 15$, one of them is special. You have submitted an ordered 4-tuple of subsets of $0, 1, 2, \ldots, 15$. Here is your reply from the grader.

1	2	3	4
Y/N	Y/N	Y/N	Y/N

What is the special number?

L5.2) In this problem, the symbol 0 represents the number zero, the symbol 1 represents the number seven, the symbol 2 represents the number five, the symbol 3 represents the number three, the symbol 4 represents the number four, the symbol 5 represents the number two, the symbol 6 represents the number nine, the symbol 7 represents the number one, the symbol 8 represents an arbitrarily large positive integer, the symbol 9 represents the number six, and the symbol ∞ represents the number eight. Compute the value of $\left|0 - 1 + 2 - 3^4 - 5 + 6 - 7^8 \times 9 - \infty\right|$.

L5.3) Let $A(x) = \lfloor \frac{x^2 - 20x + 16}{4} \rfloor$, $B(x) = \sin\left(e^{\cos\sqrt{x^2+2x+2}}\right)$, $C(x) = x^3 - 6x^2 + 5x + 15$, $H(x) = x^4 + 2x^3 + 3x^2 + 4x + 5$, $M(x) = \frac{x}{2} - 2\lfloor \frac{x}{2} \rfloor + \frac{x}{2^2} + \frac{x}{2^3} + \frac{x}{2^4} + \ldots$, $N(x) =$ the number of integers that divide $\lfloor x \rfloor$, $O(x) = |x| \log|x| \log\log|x|$, $T(x) = \sum_{n=1}^{\infty} \frac{n^x}{(n!)^3}$, and $Z(x) = \frac{x^{21}}{2016 + 20x^{16} + 16x^{20}}$ for any real number x such that the functions are defined. Determine

$$C(C(A(M(A(T(H(B(O(N(A(N(Z(A(2016)))))))))))))).$$

L5.4) In the game of Colonel Blotto, you have 100 troops to distribute among 10 castles. Submit a 10-tuple $(x_1, x_2, \ldots x_{10})$ of nonnegative integers such that $x_1 + x_2 + \ldots + x_{10} = 100$, where each x_i represent the number of troops you want to send to castle i. Your troop distribution will be matched up against each opponent's and you will win 10 points for each castle that you send more troops to (if you send the same number, you get 5 points, and if you send fewer, you get none). Your aim is to score the most points possible averaged over all opponents.

For example, if team A submits $(90, 10, 0, \ldots, 0)$, team B submits $(11, 11, 11, 11, 11, 11, 11, 11, 11, 1)$, and team C submits $(10, 10, 10, \ldots 10)$, then team A will win 10 points against team B and 15 points against team C, while team B wins 90 points against team C. Team A averages 12.5 points, team B averages 90 points, and team C averages 47.5 points.