Individual Round

CCA Math Bonanza

20 Jan 2018

I1) What is the tens digit of the sum

$$(1!)^2 + (2!)^2 + (3!)^2 + \ldots + (2018!)^2?$$

I2) Let P be the product of the first 50 nonzero square numbers. Find the largest integer k such that 7^k divides P.

I3) A Louis Vuitton store in Shanghai had a number of pairs of sunglasses which cost an average of \$900 per pair. LiAngelo Ball stole a pair which cost \$2000. Afterwards, the average cost of sunglasses in the store dropped to \$890 per pair. How many pairs of sunglasses were in the store before LiAngelo Ball stole?

I4) Zadam Heng bets Saniel Dun that he can win in a free throw contest. Zadam shoots until he has made 5 shots. He wins if this takes him 9 or fewer attempts. The probability that Zadam makes any given attempt is $\frac{1}{2}$. What is the probability that Zadam Heng wins the bet?

I5) Determine all positive numbers x such that

$$\frac{16}{x+2} + \frac{4}{x+0} + \frac{9}{x+1} + \frac{100}{x+8} = 19.$$

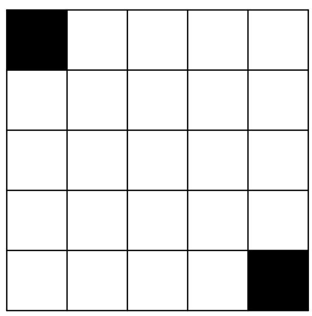
I6) A lumberjack is building a non-degenerate triangle out of logs. Two sides of the triangle have lengths $\log 101$ and $\log 2018$. The last side of his triangle has side length $\log n$, where n is an integer. How many possible values are there for n?

I7) Find all values of a such that the two polynomials

$$x^2 + ax - 1 \qquad \text{and} \qquad x^2 - x + a$$

share at least 1 root.

I8) The New York Times Mini Crossword is a 5×5 grid with the top left and bottom right corners shaded. Each row and column has a clue given (so that there are 10 clues total). Jeffrey has a $\frac{1}{2}$ chance of knowing the answer to each clue. What is the probability that he can fill in every unshaded square in the crossword?



I9) What is the area of the smallest possible square that can be drawn around a regular hexagon of side length 2 such that the hexagon is contained entirely within the square?

I10) In the land of Chaina, people pay each other in the form of links from chains. Fiona, originating from Chaina, has an open chain with 2018 links. In order to pay for things, she decides to break up the chain by choosing a number of links and cutting them out one by one, each time creating 2 or 3 new chains. For example, if she cuts the 1111th link out of her chain first, then she will have 3 chains, of lengths 1110, 1, and 907. What is the least number of links she needs to remove in order to be able to pay for anything costing from 1 to 2018 links using some combination of her chains?

I11) Square $ABCD$ has side length 1; circle Γ is centered at A with radius 1. Let M be the midpoint of BC, and let N be the point on segment CD such that MN is tangent to Γ. Compute MN.

I12) For how many integers $n \neq 1$ does $(n-1)^3$ divide $n^{2018(n-1)} - 1$?

I13) $P(x)$ is a polynomial of degree at most 6 such that such that $P(1)$, $P(2)$, $P(3)$, $P(4)$, $P(5)$, $P(6)$, and $P(7)$ are 1, 2, 3, 4, 5, 6, and 7 in some order. What is the maximum possible value of $P(8)$?

I14) Brian starts at the point $(1, 0)$ in the plane. Every second, he performs one of two moves: he can move from (a, b) to $(a - b, a + b)$ or from (a, b) to $(2a - b, a + 2b)$. How many different paths can he take to end up at $(28, -96)$?

I15) In a triangle ABC, let the B-excircle touch CA at E, C-excircle touch AB at F. If M is the midpoint of BC, then let the angle bisector of $\angle BAC$ meet BC, EF, ME, MF at D, P, E', F'. Suppose that the circumcircles of $\triangle EPE'$ and $\triangle FPF'$ meet again at a point Q and the circumcircle of $\triangle DPQ$ meets line EF again at X. If $BC = 10, CA = 20, AB = 18$, compute $|XE - XF|$.