Lightning Round

CCA Math Bonanza

20 Jan 2018

Set 1

Each question in this set is worth 1.5 points.

L1.1) Let $A = 1, B = 2, \ldots, Z = 26$. Compute $BONANZA$, where the result is the product of the numbers represented by each letter.

L1.2) The CCA Math Banana$^{\text{TM}}$ costs \$100. The cost rises 10% then drops 10%. Now what is the cost of the CCA Math Banana$^{\text{TM}}$?

L1.3) $ABCDEF$ is a hexagon inscribed in a circle such that the measure of $\angle ACE$ is 90°. What is the average of the measures, in degrees, of $\angle ABC$ and $\angle CDE$?

L1.4) What is the sum of all distinct values of x that satisfy $x^4 - x^3 - 7x^2 + 13x - 6 = 0$?

Set 2

Each question in this set is worth 1.75 points.

L2.1) Let S be the set of the first 2018 positive integers, and let T be the set of all distinct numbers of the form ab, where a and b are distinct members of S. What is the 2018th smallest member of T?

L2.2) Points X, Y, Z lie on a line in this order and point P lies off this line such that $\angle XPY = \angle PZY$. If $XY = 4$ and $YZ = 5$, compute PX.

L2.3) On January 20, 2018, Sally notices that her 7 children have ages which sum to a perfect square: their ages are 1, 3, 5, 7, 9, 11, and 13, with $1+3+5+7+9+11+13 = 49$. Let N be the age of the youngest child the next year the sum of the 7 children's ages is a perfect square on January 20th, and let P be that perfect square. Find $N + P$.

L2.4) Alex, Bertha, Cameron, Dylan, and Ellen each have a different toy. Each kid puts each of their own toys into a large bag. The toys are then randomly distributed such that each kid receives a toy. How many ways are there for exactly one kid to get the same toy that they put in?

Set 3

Each question in this set is worth 2.25 points.

L3.1) The number $16^4 + 16^2 + 1$ is divisible by four distinct prime numbers. Compute the sum of these four primes.

L3.2) How many positive integers $n \leq 100$ satisfy $\lfloor n\pi \rfloor = \lfloor (n-1)\pi \rfloor + 3$? Here $\lfloor x \rfloor$ is the greatest integer less than or equal to x; for example, $\lfloor \pi \rfloor = 3$.

L3.3) On January 15 in the stormy town of Stormville, there is a 50% chance of rain. Every day, the probability of it raining has a 50% chance of being $\frac{2017}{2016}$ times that of the previous day (or 100% if this new quantity is over 100%) and a 50% chance of being $\frac{1007}{2016}$ times that of the previous day. What is the probability that it rains on January 20?

L3.4) Consider equilateral triangle ABC with side length 1. Suppose that a point P in the plane of the triangle satisfies

$$2AP = 3BP = 3CP = \kappa$$

for some constant κ. Compute the sum of all possible values of κ.

Set 4

Each question in this set is worth 2.5 points.

L4.1) Let S be the set of all ordered triples (a, b, c) of positive integers such that $(b-c)^2 + (c-a)^2 + (a-b)^2 = 2018$ and $a + b + c \leq M$ for some positive integer M. Given that $\sum_{(a,b,c)\in S} a = k$, what is

$$\sum_{(a,b,c)\in S} a\left(a^2 - bc\right)$$

in terms of k?

L4.2) A subset of $\{1, 2, 3, \ldots, 2017, 2018\}$ has the property that none of its members are 5 times another. What is the maximum number of elements that such a subset could have?

L4.3) ABC is an isosceles triangle with $AB = AC$. Point D is constructed on AB such that $\angle BCD = 15^\circ$. Given that $BC = \sqrt{6}$ and $AD = 1$, find the length of CD.

L4.4) Alice and Billy are playing a game on a number line. They both start at 0. Each turn, Alice has a $\frac{1}{2}$ chance of moving 1 unit in the positive direction, and a $\frac{1}{2}$ chance of moving 1 unit in the negative direction, while Billy has a $\frac{2}{3}$ chance of moving 1 unit in the positive direction, and a $\frac{1}{3}$ chance of moving 1 unit in the negative direction. Alice and Billy alternate turns, with Alice going first. If a player reaches 2, they win and the game ends, but if they reach -2, they lose and the other player wins, and the game ends. What is the probability that Billy wins?

Set 5

Each question in this set is worth 2 points.

L5.1) Estimate the number of five-card combinations from a standard 52-card deck that contain a pair (two cards with the same number).

An estimate of E earns $2e^{-\frac{|A-E|}{20000}}$ points, where A is the actual answer.

L5.2) Two circles of equal radii are drawn to intersect at X and Y. Suppose that the two circles bisect each other's areas. If the measure of minor arc $\widehat{XY}$ is θ degrees, estimate $\lfloor 1000\theta \rfloor$.

An estimate of E earns $2e^{-\frac{|A-E|}{50000}}$ points, where A is the actual answer.

L5.3) Choose an integer n from 1 to 10 inclusive as your answer to this problem. Let m be the number of distinct values in $\{1, 2, \ldots, 10\}$ chosen by all teams at the Math Bonanza for this problem which are greater than or equal to n. Your score on this problem will be $\frac{mn}{15}$. For example, if 5 teams choose 1, 2 teams choose 2, and 6 teams choose 3 with these being the only values chosen, and you choose 2, you will receive $\frac{4}{15}$ points.

L5.4) Welcome to the **USAYNO**, a twelve-part question where each part has a yes/no answer. If you provide C correct answers, your score on this problem will be $\frac{C}{6}$.

Your answer should be a twelve-character string containing 'Y' (for yes) and 'N' (for no). For instance if you think a, c, and f are 'yes' and the rest are 'no', you should answer YNYNNYNNNNNN.

(a) Is there a positive integer n such that the sum of the digits of $2018n + 1337$ in base 10 is 2018 more than the sum of the digits of $2018n + 1337$ in base 4?

(b) Is there a fixed constant θ such that for all triangles ABC with

$$2018AB^2 = 2018CA^2 + 2017CA \cdot CB + 2018CB^2,$$

one of the angles of ABC is θ?

(c) Adam lists out every possible way to arrange the letters of "CCACCACCA" (including the given arrangement) at 1 arrangement every 5 seconds. Madam lists out every possible way to arrange the letters of "CCACCAA" (including the given arrangement) at 1 arrangement every 12 seconds. Does Adam finish first?

(d) Do there exist real numbers a, b, c, none of which is the average of the other two, such that

$$\frac{1}{b+c-2a} + \frac{1}{c+a-2b} + \frac{1}{a+b-2c} = 0?$$

(e) Let $f(x) = \frac{2^x - 2}{x} - 1$. Is there an integer n such that

$$f(n), f(f(n)), f(f(f(n))), \ldots$$

are all integers?

(f) In an elementary school with 2585 students and 159 classes (every student is in exactly one class), each student reports the size of their class. The principal of the school takes the average of all of these numbers and calls it X. The principal then computes the average size of each class and calls it Y. Is it necessarily true that $X > Y$?

(g) Six sticks of lengths 3, 5, 7, 11, 13, and 17 are put together to form a hexagon. From a point inside the hexagon, a circular water balloon begins to expand and will stop expanding once it hits any stick. Is it possible that once the balloon stops expanding, it is touching each of the six sticks?

(h) A coin is biased so that it flips heads and tails (and only heads or tails) each with a positive rational probability (not necessarily $\frac{1}{2}$). Is it possible that on average, it takes exactly twice as long to flip two heads in a row as it is to flip two tails in a row?

(i) Does there exist a base b such that 2018_b is prime?

(j) Does there exist a sequence of 2018 distinct real numbers such that no 45 terms (not necessarily consecutive) can be examined, in order, and be in strictly increasing or strictly decreasing order?

(k) Does there exist a scalene triangle ABC such that there exist two distinct rectangles $PQRS$ inscribed in $\triangle ABC$ with $P \in AB$, $Q, R \in BC$, $S \in AC$ such that the angle bisectors of $\angle PAS$, $\angle PQR$, and $\angle SRQ$ concur?

(l) For three vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ with $\mathbf{u}_i = (x_{i,1}, x_{i,2}, x_{i,3}, x_{i,4})$, define

$$f(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = 1 - \prod_{j=1}^{4} \left(1 + (x_{2,j} - x_{3,j})^2 + (x_{3,j} - x_{1,j})^2 + (x_{1,j} - x_{2,j})^2\right).$$

Are there any sequences $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{18}$ of distinct vectors with four components, with all components in $\{1, 2, 3\}$, such that

$$\prod_{1 \le i < j < k \le 18} f(\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k) \equiv 1 \pmod{3}?$$