Individual Round

CCA Math Bonanza

2 Feb 2019

I1) Consider the arithmetic sequence of integers with first term -7 and second term 17. What is the sum of the squares of the first three terms of the sequence?

I2) Square 1 is drawn with side length 4. Square 2 is then drawn inside of Square 1, with its vertices at the midpoints of the sides of Square 1. Given Square n for a positive integer n, we draw Square $n+1$ with vertices at the midpoints of the sides of Square n. For any positive integer n, we draw Circle n through the four vertices of Square n. What is the area of Circle 7?

I3) Sristan Thin is walking around the Cartesian plane. From any point (x, y), Sristan can move to $(x+1, y)$ or $(x+1, y+3)$. How many paths can Sristan take from $(0,0)$ to $(9,9)$?

I4) How many ordered pairs (a, b) of positive integers are there such that

$$\gcd(a, b)^3 = \operatorname{lcm}(a, b)^2 = 4^6$$

is true?

I5) How many ways are there to rearrange the letters of CCAMB such that at least one C comes before the A?

I6) If distinct digits D, E, L, M, Q (between 0 and 9 inclusive) satisfy

$$\begin{array}{r} E\ L \\ +\,M\,E\,M \\ \hline Q\ E\ D \end{array}$$

what is the maximum possible value of the three digit integer QED?

I7) How many permutations π of $\{1, 2, \ldots, 7\}$ are there such that $\pi(k) \leq 2k$ for $k = 1, \ldots, 7$? A permutation π of a set S is a function from S to itself such that if $a \neq b$, then $\pi(a) \neq \pi(b)$. For example, $\pi(x) = x$ and $\pi(x) = 8 - x$ are permutations of $\{1, 2, \ldots, 7\}$ but $\pi(x) = 1$ is not.

I8) If $a! + (a+2)!$ divides $(a+4)!$ for some nonnegative integer a, what are all possible values of a?

I9) Isosceles triangle $\triangle ABC$ has $\angle BAC = \angle ABC = 30^\circ$ and $AC = BC = 2$. If the midpoints of BC and AC are M and N, respectively, and the circumcircle of $\triangle CMN$ meets AB at D and E with D closer to A than E is, what is the area of $MNDE$?

I10) What is the minimum possible value of

$$|x| - |x-1| + |x+2| - |x-3| + |x+4| - \cdots - |x-2019|$$

over all real x?

I11) Let G be the centroid of triangle ABC with $AB = 13, BC = 14, CA = 15$. Calculate the sum of the distances from G to the three sides of the triangle.

Note: The *centroid* of a triangle is the point that lies on each of the three line segments between a vertex and the midpoint of its opposite side.

I12) Let $f(x, y) = x^2\left((x+2y)^2 - y^2 + x - 1\right)$. If $f(a, b+c) = f(b, c+a) = f(c, a+b)$ for distinct numbers a, b, c, what are all possible values of $a+b+c$?

I13) Convex quadrilateral $ABCD$ has $AB = 20$, $BC = CD = 26$, and $\angle ABC = 90°$. Point P is on DA such that $\angle PBA = \angle ADB$. If $PB = 20$, compute the area of $ABCD$.

I14) Call an odd prime p *adjective* if there exists an infinite sequence $a_0, a_1, a_2, \ldots$ of positive integers such that

$$a_0 \equiv 1 + \frac{1}{a_1} \equiv 1 + \frac{1}{1 + \frac{1}{a_2}} \equiv 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{a_3}}} \equiv \ldots \pmod{p}.$$

What is the sum of the first three odd primes that are *not* adjective?

Note: For two common fractions $\frac{a}{b}$ and $\frac{c}{d}$, we say that $\frac{a}{b} \equiv \frac{c}{d} \pmod{p}$ if p divides $ad - bc$ and p does not divide bd.

I15) Before Harry Potter died, he decided to bury his wand in one of eight possible locations (uniformly at random). A squad of Death Eaters decided to go hunting for the wand. They know the eight locations but have poor vision, so even if they're at the correct location they only have a 50% chance of seeing the wand. They also get tired easily, so they can only check three different locations a day. At least they have one thing going for them: they're clever. Assuming they strategize optimally, what is the expected number of days it will take for them to find the wand?