Individual Round

CCA Math Bonanza

18 Jan 2020

I1) An ant is crawling along the coordinate plane. Each move, it moves one unit up, down, left, or right with equal probability. If it starts at $(0, 0)$, what is the probability that it will be at either $(2, 1)$ or $(1, 2)$ after 6 moves?

I2) Circles ω and γ are drawn such that ω is internally tangent to γ, the distance between their centers are 5, and the area inside of γ but outside of ω is 100π. What is the sum of the radii of the circles?

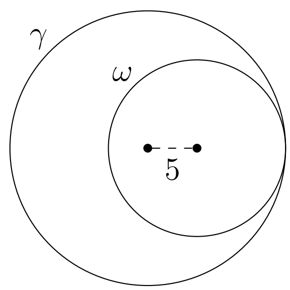

Figure not drawn to scale

I3) Compute the remainder when $\left(\frac{2^5}{2}\right)^5$ is divided by 5.

I4) Alan, Jason, and Shervin are playing a game with MafsCounts questions. They each start with 2 tokens. In each round, they are given the same MafsCounts question. The first person to solve the MafsCounts question wins the round and steals one token from each of the other players in the game. They all have the same probability of winning any given round. If a player runs out of tokens, they are removed from the game. The last player remaining wins the game.

If Alan wins the first round but does not win the second round, what is the probability that he wins the game?

I5) Let $f(x) = x^2 - kx + (k-1)^2$ for some constant k. What is the largest possible real value of k such that f has at least one real root?

I6) Let P be a point outside a circle Γ centered at point O, and let PA and PB be tangent lines to circle Γ. Let segment PO intersect circle Γ at C. A tangent to circle Γ through C intersects PA and PB at points E and F, respectively. Given that $EF = 8$ and $\angle APB = 60^\circ$, compute the area of $\triangle AOC$.

I7) Define the binary operation $a\Delta b = ab + a - 1$. Compute

$$10\Delta(10\Delta(10\Delta(10\Delta(10\Delta(10\Delta(10\Delta(10\Delta(10\Delta10))))))))$$

where 10 is written 10 times.

I8) Compute the remainder when the largest integer below $\frac{3^{123}}{5}$ is divided by 16.

I9) A sequence a_n of real numbers satisfies $a_1 = 1$, $a_2 = 0$, and $a_n = (S_{n-1} + 1)S_{n-2}$ for all integers $n \geq 3$, where $S_k = a_1 + a_2 + \cdots + a_k$ for positive integers k. What is the smallest integer $m > 2$ such that 127 divides a_m?

I10) Annie takes a 6 question test, with each question having two parts each worth 1 point. On each **part**, she receives one of nine letter grades {A,B,C,D,E,F,G,H,I} that correspond to a unique numerical score. For each **question**, she receives the sum of her numerical scores on both parts. She knows that A corresponds to 1, E corresponds to 0.5, and I corresponds to 0.

When she receives her test, she realizes that she got two of each of A, E, and I, and she is able to determine the numerical score corresponding to all 9 markings. If n is the number of ways she can receive letter grades, what is the exponent of 2 in the prime factorization of n?

I11) Points C, A, D, M, E, B, F lie on a line in that order such that $CA = AD = EB = BF = 1$ and M is the midpoint of DB. Let X be a point such that a quarter circle arc exists with center D and endpoints C, X. Suppose that line XM is tangent to the unit circle centered at B. Compute AB.

I12) Find all pairs (a, b) of positive integers satisfying the following conditions:

- $a \leq b$
- ab is a perfect cube
- No divisor of a or b is a perfect cube greater than 1
- $a^2 + b^2 = 85\,\mathrm{lcm}(a, b)$

I13) Let n be a positive integer. Compute, in terms of n, the number of sequences $(x_1, \ldots, x_{2n})$ with each $x_i \in \{0, 1, 2, 3, 4\}$ such that $x_1^2 + \cdots + x_{2n}^2$ is divisible by 5.

I14) An ant starts at the point $(0, 0)$ in the coordinate plane. It can make moves from lattice point (x_1, y_1) to lattice point (x_2, y_2) whenever $x_2 \geq x_1$, $y_2 \geq y_1$, and $(x_1, y_1) \neq (x_2, y_2)$. For all nonnegative integers m, n, define $a_{m,n}$ to be the number of possible sequences of moves from $(0, 0)$ to (m, n) (e.g. $a_{0,0} = 1$ and $a_{1,1} = 3$). Compute

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_{m,n}}{10^{m+n}}.$$

I15) Let θ be an obtuse angle with $\sin\theta = \frac{3}{5}$. If an ant starts at the origin and repeatedly moves 1 unit and turns by an angle of θ, there exists a region R in the plane such that for every point $P \in R$ and every constant $c > 0$, the ant is within a distance c of P at some point in time (so the ant gets arbitrarily close to every point in the set). What is the largest possible area of R?