Lightning Round

CCA Math Bonanza

18 Jan 2020

Set 1

Each question in this set is worth 1.5 *points.*

L1.1) We know that 201 and 9 give the same remainder when divided by 24. What is the smallest positive integer k such that $201 + k$ and $9 + k$ give the same remainder when divided by 24?

L1.2) Let $a_1 = 3$, $a_2 = 7$, and $a_3 = 1$. Let $b_0 = 0$ and for all positive integers n, let $b_n = 10b_{n-1} + a_n$. Compute $b_1 \times b_2 \times b_3$.

L1.3) If $ABCDE$ is a regular pentagon and X is a point in its interior such that CDX is equilateral, compute $\angle AXE$ in degrees.

L1.4) Let ABC be a triangle with $AB = 3$, $BC = 4$, and $CA = 5$. Points A_1, B_1, and C_1 are chosen on its incircle. Compute the maximum possible sum of the areas of triangles A_1BC, AB_1C, and ABC_1.

Set 2

Each question in this set is worth 1.75 *points.*

L2.1) We know that 201 and 9 give the same remainder when divided by 24. What is the smallest positive integer k such that $201 + k$ and $9 + k$ give the same remainder when divided by $24 + k$?

L2.2) A rectangular box with side lengths 1, 2, and 16 is cut into two congruent smaller boxes with integer side lengths. Compute the square of the largest possible length of the space diagonal of one of the smaller boxes.

L2.3) 3 uncoordinated aliens launch a 3-day attack on 4 galaxies. Each day, each of the three aliens chooses a galaxy uniformly at random from the remaining galaxies and destroys it. They make their choice simultaneously and independently, so two aliens could destroy the same galaxy. If the probability that every galaxy is destroyed by the end of the attack can be expressed as $\frac{m}{n}$ for relatively prime positive integers m, n, what is $m + n$?

L2.4) If

$$\sum_{k=1}^{1000} \left(\frac{k+1}{k} + \frac{k}{k+1} \right) = \frac{m}{n}$$

for relatively prime positive integers m, n, compute $m + n$.

Set 3

Each question in this set is worth 2.25 points.

L3.1) For some positive integer n, the sum of all odd positive integers between $n^2 - n$ and $n^2 + n$ is a number between 9000 and 10000, inclusive. Compute n.

L3.2) Archit and Ayush are walking around on the set of points (x, y) for all integers $-1 \leq x, y \leq 1$. Archit starts at $(1, 1)$ and Ayush starts at $(1, 0)$. Each second, they move to another point in the set chosen uniformly at random among the points with distance 1 away from them. If the probability that Archit goes to the point $(0, 0)$ strictly before Ayush does can be expressed as $\frac{m}{n}$ for relatively prime positive integers m, n, compute $m + n$.

L3.3) Compute the largest prime factor of $111^2 + 11^3 + 1^1$.

L3.4) Willy Wonka has n distinguishable pieces of candy that he wants to split into groups. If the number of ways for him to do this is $p(n)$, then we have

n	1	2	3	4	5	6	7	8	9	10
$p(n)$	1	2	5	15	52	203	877	4140	21147	115975

Define a *splitting* of the n distinguishable pieces of candy to be a way of splitting them into groups. If Willy Wonka has 8 candies, what is the sum of the number of groups over all splittings he can use?

Set 4

Each question in this set is worth 2.5 points.

L4.1) Alice picks a number uniformly at random from the first 5 even positive integers, and Palice picks a number uniformly at random from the first 5 odd positive integers. If Alice picks a larger number than Palice with probability $\frac{m}{n}$ for relatively prime positive integers m, n, compute $m + n$.

L4.2) Let $a_0, a_1, \ldots$ be a sequence of positive integers such that $a_0 = 1$, and for all positive integers n, a_n is the smallest composite number relatively prime to all of $a_0, a_1, \ldots, a_{n-1}$. Compute a_{10}.

L4.3) Let $ABCD$ be a convex quadrilateral such that $AB = 4$, $BC = 5$, $CA = 6$, and $\triangle ABC$ is similar to $\triangle ACD$. Let P be a point on the extension of DA past A such that $\angle BDC = \angle ACP$. Compute DP^2.

L4.4) A sequence $\{a_n\}$ is defined such that $a_i = i$ for $i = 1, 2, 3 \ldots, 2020$ and for $i > 2020$, a_i is the average of the previous 2020 terms. What is the largest integer less than or equal to $\lim\limits_{n \to \infty} a_n$?

Set 5

Each question in this set is worth 2 *points.*

L5.1) Professor Shian Bray is buying CCA Math Bananas$^{\text{TM}}$. He starts with \$500. The first CCA Math Banana$^{\text{TM}}$ he buys costs \$1. Each time after he buys a CCA Math Banana$^{\text{TM}}$, the cost of a CCA Math Banana$^{\text{TM}}$ doubles with probability $\frac{1}{2}$ (otherwise staying the same). Professor Bray buys CCA Math Bananas$^{\text{TM}}$ until he cannot afford any more, ending with n CCA Math Bananas$^{\text{TM}}$. Estimate the expected value of n. An estimate of E earns $2^{1-0.25|E-A|}$ points, where A is the actual answer.

L5.2) A teacher writes the positive integers from 1 to 12 on a blackboard. Every minute, they choose a number k uniformly at random from the written numbers, subtract k from each number $n \geq k$ on the blackboard (without touching the numbers $n < k$), and erase every 0 on the board. Estimate the expected number of minutes that pass before the board is empty. An estimate of E earns $2^{1-0.5|E-A|}$ points, where A is the actual answer.

L5.3) Estimate the number of pairs of integers $1 \leq a, b \leq 1000$ satisfying $\gcd(a, b) = \gcd(a+1, b+1)$. An estimate of E earns $2^{1-0.00002|E-A|}$ points, where A is the actual answer.

L5.4) Submit a positive integer less than or equal to 15. Your goal is to submit a number that is close to the number of teams submitting it. If you submit N and the total number of teams at the competition (including your own team) who submit N is T, your score will be $\frac{2}{0.5|N-T|+1}$.