Lightning Round

CCA Math Bonanza

17 Apr 2021

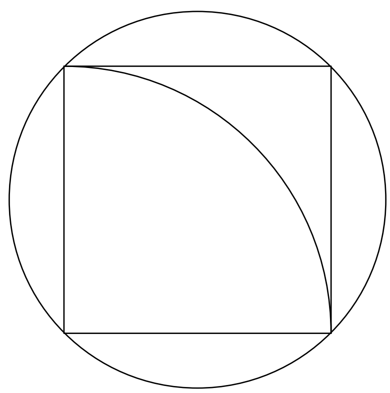
Set 1

Each question in this set is worth 1.5 points.

L1.1) Compute

$$(2 + 0 \cdot 2 \cdot 1) + (2 + 0 - 2) \cdot (1) + (2 + 0) \cdot (2 - 1) + (2) \cdot \left(0 + 2^{-1}\right).$$

L1.2) A square is inscribed in a circle of radius 6. A quarter circle is inscribed in the square, as shown in the diagram below. Given the area of the region inside the circle but outside the quarter circle is $n\pi$ for some positive integer n, what is n?



L1.3) A coin is flipped 20 times. Let p be the probability that each of the following sequences of flips occur exactly twice:

- one head, two tails, one head
- one head, one tails, two heads.

Given that p can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers, compute $\gcd(m, n)$.

L1.4) On Day 1, Alice starts with the number $a_1 = 5$. For all positive integers $n > 1$, on Day n, Alice randomly selects a positive integer a_n between a_{n-1} and $2a_{n-1}$, inclusive. Given that the probability that all of $a_2, a_3, \ldots, a_7$ are odd can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers, compute $m + n$.

Set 2

Each question in this set is worth 1.75 *points.*

L2.1) Let ABC be a triangle with $AB = 3$, $BC = 4$, and $CA = 5$. The line through A perpendicular to AC intersects line BC at D, and the line through C perpendicular to AC intersects line AB at E. Compute the area of triangle BDE.

L2.2) Given that nonzero reals a, b, c, d satisfy $a^b = c^d$ and $\frac{a}{2c} = \frac{b}{d} = 2$, compute $\frac{1}{c}$.

L2.3) Broady The Boar is playing a boring board game consisting of a circle with 2021 points on it, labeled 0, 1, 2, ... 2020 in that order clockwise. Broady is rolling 2020-sided die which randomly produces a whole number between 1 and 2020, inclusive.

Broady starts at the point labelled 0. After each dice roll, Broady moves up the same number of points as the number rolled (point 2020 is followed by point 0). For example, if they are at 0 and roll a 5, they end up at 5. If they are at 2019 and roll a 3, they end up at 1.

Broady continues rolling until they return to the point labelled 0. What is the expected number of times they roll the dice?

L2.4) Compute the number of two digit positive integers that are divisible by both of their digits. For example, 36 is one of these two digit positive integers because it is divisible by both 3 and 6.

Set 3

Each question in this set is worth 2.25 *points.*

L3.1) A point is chosen uniformly at random from the interior of a unit square. Let p be the probability that any circle centered at the point that intersects a diagonal of the square must also intersect a side of the square. Given that p^2 can be written as $m - \sqrt{n}$ for positive integers m and n, what is $m + n$?

L3.2) A frog is standing in a center of a 3×3 grid of lilypads. Each minute, the frog chooses a square that shares exactly one side with their current square uniformly at random, and jumps onto the lilypad on their chosen square. The frog stops jumping once it reaches a lilypad on a corner of the grid. What is the expected number of times the frog jumps?

L3.3) Compute the smallest positive integer that gives a remainder of 1 when divided by 11, a remainder of 2 when divided by 21, and a remainder of 5 when divided by 51.

L3.4) Compute the sum of $x^2 + y^2$ over all four ordered pairs (x, y) of real numbers satisfying $x = y^2 - 20$ and $y = x^2 + x - 21$.

Set 4

Each question in this set is worth 2.5 *points.*

L4.1) Suppose that x^2+px+q has two distinct roots $x=a$ and $x=b$. Furthermore, suppose that the positive difference between the roots of x^2+ax+b, the positive difference between the roots of x^2+bx+a, and twice the positive difference between the roots of x^2+px+q are all equal. Given that q can be expressed in the form $\frac{m}{m}$, where m and n are relatively prime positive integers, compute $m+n$.

L4.2) Compute the number of (not necessarily convex) polygons in the coordinate plane with‘ the following properties:

- If the coordinates of a vertex are (x, y), then x, y are integers and $1 \leq |x|+|y| \leq 3$
- Every side of the polygon is parallel to either the x or y axis
- The point $(0, 0)$ is contained in the interior of the polygon.

L4.3) For a positive integer n, let $f(n)$ be the sum of the positive integers that divide at least one of the nonzero base 10 digits of n. For example, $f(96) = 1+2+3+6+9 = 21$. Find the largest positive integer n such that for all positive integers k, there is some positive integer a such that $f^k(a) = n$, where $f^k(a)$ denotes f applied k times to a.

L4.4) Let ABC be a triangle with $AB = 5$, $BC = 7$, $CA = 8$, and let M be the midpoint of BC. Points P and Q are chosen on the circumcircle of ABC such that MPQ and ABC are similar (with vertices in that order). The product of all different possible areas of MPQ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m+n$.

Set 5

Each question in this set is worth 2 *points.*

L5.1) Estimate the number of distinct submissions to this problem. Your submission must be a positive integer less than or equal to 50. If you submit E, and the actual number of distinct submissions is D, you will receive a score of $\frac{2}{0.5|E-D|+1}$.

L5.2) Define the sequences $x_0, x_1, x_2, \ldots$ and $y_0, y_1, y_2, \ldots$ such that $x_0 = 1$, $y_0 = 2021$, and for all nonnegative integers n, we have $x_{n+1} = \sqrt{x_n y_n}$ and $y_{n+1} = \frac{x_n + y_n}{2}$. There is some constant X such that as n grows large, $x_n - X$ and $y_n - X$ both approach 0. Estimate X.

An estimate of E earns $\max(0, 2 - 0.02|A - E|)$ points, where A is the actual answer.

L5.3) Let N be the number of sequences of words (not necessarily grammatically correct) that have the property that the first word has one letter, each word can be obtained by inserting a letter somewhere in the previous word, and the final word is CCAMATH-BONANZA. Here is an example of a possible sequence:

N, NA, NZA, BNZA, BNAZA, BONAZA, BONANZA, CBONANZA, CABONANZA, CAMBONANZA, CAMABONANZA, CAMAHBONANZA, CCAMAHBONANZA, CCAMATHBONANZA.

Estimate $\frac{N}{12!}$. An estimate of $E > 0$ earns $\max(0, 4 - 2\max(A/E, E/A))$ points, where A is the actual answer. An estimate of $E = 0$ earns 0 points.

L5.4) Estimate the number of primes among the first thousand primes divide some term of the sequence

$$2^0 + 1, 2^1 + 1, 2^2 + 1, 2^3 + 1, \ldots.$$

An estimate of E earns $2^{1-0.02|A-E|}$ points, where A is the actual answer.