# Team Round

## CCA Math Bonanza

## 17 Apr 2021

T1) How many sequences of words (not necessarily grammatically correct) have the property that the first word has one letter, each word can be obtained by inserting a letter somewhere in the previous word, and the final word is CCAMT? Here are examples of possible sequences:

C,CA,CAM,CCAM,CCAMT.

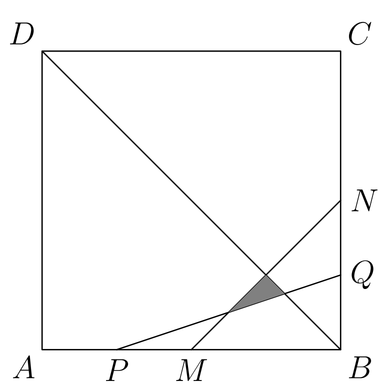
A,AT,CAT,CAMT,CCAMT.

T2) Given that real numbers $a$, $b$, and $c$ satisfy $ab = 3$, $ac = 4$, and $b + c = 5$, the value of $bc$ can be written as $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Compute $m + n$.

T3) For any real number $x$, we let $\lfloor x \rfloor$ be the unique integer $n$ such that $n \leq x < n + 1$. For example. $\lfloor 31.415 \rfloor = 31$. Compute

$$2020^{2021} - \left\lfloor \frac{2020^{2021}}{2021} \right\rfloor (2021).$$

T4) Let $ABCD$ be a unit square. Points $M$ and $N$ are the midpoints of sides $AB$ and $BC$ respectively. Let $P$ and $Q$ be the midpoints of line segments $AM$ and $BN$ respectively. Find the reciprocal of the area of the triangle enclosed by the three line segments $PQ$, $MN$, and $DB$.



T5) We say that a *special word* is any sequence of letters **containing a vowel**. How many ordered triples of special words $(W_1, W_2, W_3)$ have the property that if you concatenate the three words, you obtain a rearrangement of “aadvarks”?

For example, the number of triples of special words such that the concatenation is a rearrangement of “adaa” is 6, and all of the possible triples are:

(da,a,a),(ad,a,a),(a,da,a),(a,ad,a),(a,a,da),(a,a,ad).

T6) Three spheres have radii 144, 225, and 400, are pairwise externally tangent to each other, and are all tangent to the same plane at $A$, $B$, and $C$. Compute the area of triangle $ABC$.

T7) Find the sum of all positive integers $n$ with the following properties:

- $n$ is not divisible by any primes larger than 10.
- For some positive integer $k$, the positive divisors of $n$ are

$$1 = d_1 < d_2 < d_3 \cdots < d_{2k} = n.$$

- The divisors of $n$ have the property that

$$d_1 + d_2 + \cdots + d_k = 3k.$$

T8) Let $ABC$ be a triangle with $AB = 9$ and $AC = 12$. Point $B'$ is chosen on line $AC$ such that the midpoint of $B$ and $B'$ is equidistant from $A$ and $C$. Point $C'$ is chosen similarly. Given that the circumcircle of $AB'C'$ is tangent to $BC$, compute $BC^2$.

T9) Each number in the list $1, 2, 3, \ldots, 10$ is either colored red or blue. Numbers are colored independently, and both colors are equally probable. The expected value of the number of positive integers expressible as a sum of a red integer and a blue integer can be written as $\frac{m}{n}$ for relatively prime positive integers $m$ and $n$. What is $m + n$?

T10) Given that positive integers $a, b$ satisfy

$$\frac{1}{a + \sqrt{b}} = \sum_{i=0}^{\infty} \frac{\sin^2\left(\frac{10^\circ}{3^i}\right)}{\cos\left(\frac{30^\circ}{3^i}\right)},$$

where all angles are in degrees, compute $a + b$.