Individual Round

CCA Math Bonanza

17 Apr 2021

I1) Compute the number of positive integer divisors of 2121 with a units digit of 1.

I2) Let ABC be a triangle with $AB = 13$, $BC = 14$, and $CA = 15$. Points P, Q, and R are chosen on segments BC, CA, and AB, respectively, such that triangles AQR, BPR, CPQ have the same perimeter, which is $\frac{4}{5}$ of the perimeter of PQR. What is the perimeter of PQR?

I3) How many reorderings of $2, 3, 4, 5, 6$ have the property that every pair of adjacent numbers are relatively prime?

I4) Given that nonzero real numbers x and y satisfy $x + \frac{1}{y} = 3$ and $y + \frac{1}{x} = 4$, what is $xy + \frac{1}{xy}$?

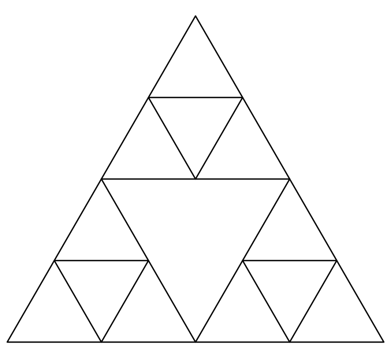
I5) If digits A, B, and C (between 0 and 9 inclusive) satisfy

$$\begin{array}{r} C\,C\,A \\ +\,B\,2\,B \\ \hline A\,8\,8 \end{array}$$

what is $A \cdot B \cdot C$?

I6) Let ABC be a right triangle with $AB = 3$, $BC = 4$, and $\angle B = 90^\circ$. Points P, Q, and R are chosen on segments AB, BC, and CA, respectively, such that PQR is an equilateral triangle, and $BP = BQ$. Given that BP can be written as $\frac{\sqrt{a}-b}{c}$, where a, b, c are positive integers and $\gcd(b, c) = 1$, what is $a + b + c$?

I7) The image below consists of a large triangle divided into 13 smaller triangles. Let N be the number of ways to color each smaller triangle one of red, green, and blue such that if T_1 and T_2 are smaller triangles whose perimeters intersect at more than one point, T_1 and T_2 have two different colors. Compute the number of positive integer divisors of N.



I8) Joel is rolling a 6-sided die. After his first roll, he can choose to re-roll the die up to 2 more times. If he rerolls strategically to maximize the expected value of the final value the die lands on, the expected value of the final value the die lands on can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$?

I9) Points A, B, C, D, and E are on the same plane such that A, E, C lie on a line in that order, B, E, D lie on a line in that order, $AE = 1$, $BE = 4$, $CE = 3$, $DE = 2$, and $\angle AEB = 60°$. Let AB and CD intersect at P. The square of the area of quadrilateral $PAED$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$?

I10) Let $\mathrm{s}_b(k)$ denote the sum of the digits of k in base b. Compute

$$\mathrm{s}_{101}(33) + \mathrm{s}_{101}(66) + \mathrm{s}_{101}(99) + \cdots + \mathrm{s}_{101}(3333).$$

I11) An triangle with coordinates (x_1, y_1), (x_2, y_2), (x_3, y_3) has centroid at $(1, 1)$. The ratio between the lengths of the sides of the triangle is $3 : 4 : 5$. Given that

$$x_1^3 + x_2^3 + x_3^3 = 3x_1x_2x_3 + 20 \quad \text{and} \quad y_1^3 + y_2^3 + y_3^3 = 3y_1y_2y_3 + 21,$$

the area of the triangle can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$?

I12) Let ABC be a triangle, let the A-altitude meet BC at D, let the B-altitude meet AC at E, and let $T \neq A$ be the point on the circumcircle of ABC such that $AT || BC$. Given that D, E, T are collinear, if $BD = 3$ and $AD = 4$, then the area of ABC can be written as $a + \sqrt{b}$, where a and b are positive integers. What is $a+b$?

I13) Find the sum of the two smallest odd primes p such that for some integers a and b, p does not divide b, b is even, and $p^2 = a^3 + b^2$.

I14) For an ordered 10-tuple of nonnegative integers $a_1, a_2, \ldots, a_{10}$, we denote

$$f(a_1, a_2, \ldots, a_{10}) = \left(\prod_{i=1}^{10} \binom{20 - (a_1 + a_2 + \cdots + a_{i-1})}{a_i} \right) \cdot \left(\sum_{i=1}^{10} \binom{18+i}{19} a_i \right).$$

When $i = 1$, we take $a_1 + a_2 + \cdots + a_{i-1}$ to be 0. Let N be the average of $f(a_1, a_2, \ldots, a_{10})$ over all 10-tuples of nonnegative integers $a_1, a_2, \ldots, a_{10}$ satisfying

$$a_1 + a_2 + \cdots + a_{10} = 20.$$

Compute the number of positive integer divisors of N.

I15) Let N be the number of functions f from $\{1, 2, \ldots, 8\}$ to $\{1, 2, 3, \ldots, 255\}$ with the property that:

- $f(k) = 1$ for some $k \in \{1, 2, 3, 4, 5, 6, 7, 8\}$
- If $f(a) = f(b)$, then $a = b$.

- For all $n \in \{1, 2, 3, 4, 5, 6, 7, 8\}$, if $f(n) \neq 1$, then $f(k) + 1 > \frac{f(n)}{2} \geq f(k)$ for some $k \in \{1, 2, \ldots, 7, 8\}$.
- For all $k, n \in \{1, 2, 3, 4, 5, 6, 7, 8\}$, if $f(n) = 2f(k) + 1$, then $k < n$.

Compute the number of positive integer divisors of N.