Individual Round

CCA Math Bonanza

23 Apr 2022

I1) Asteroids A and B have circular orbits around the same star. Asteroid A is located 400 km away from the star and takes 8000 hours to complete one full revolution. Asteroid B is located 100 km away and the speed of Asteroid B is twice the speed of Asteroid A. Find how long it takes for Asteroid B to complete one full revolution in hours.

I2) Alice, Bob, Cassie, Dave, and Ethan are going on a road trip and need to arrange themselves among a drivers seat, a passenger seat, and three distinguishable back row seats. Alice, Bob, and Cassie are not allowed to drive. Alice and Bob are also not allowed to sit in the front passenger seat. Find the number of possible seating arrangements.

I3) Let $S = 1, 2, \cdots, 100$. X is a subset of S such that no two distinct elements in X multiply to an element in X. Find the maximum number of elements of X.

I4) Burrito Bear has a white unit square. She inscribes a circle inside of the square and paints it black. She then inscribes a square inside the black circle and paints it white. Burrito repeats this process indefinitely. The total black area can be expressed as $\frac{a\pi+b}{c}$. Find $a + b + c$.

I5) Let Γ_1 be a circle with radius $\frac{5}{2}$. A, B, and C are points on Γ_1 such that $\overline{AB} = 3$ and $\overline{AC} = 5$. Let Γ_2 be a circle such that Γ_2 is tangent to AB and BC at Q and R, and Γ_2 is also internally tangent to Γ_1 at P. Γ_2 intersects AC at X and Y. $[PXY]$ can be expressed as $\frac{a\sqrt{b}}{c}$. Find $a + b + c$.

I6) Let regular tetrahedron $ABCD$ have center O. Find $\tan^2(\angle AOB)$.

I7) Let

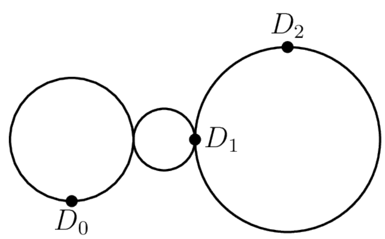
$$A = \{2, 4, \ldots, 1000\},$$
$$B = \{3, 6, \ldots, 999\},$$
$$C = \{5, 10, \ldots, 1000\},$$
$$D = \{7, 14, \ldots, 994\},$$
$$E = \{11, 22, \ldots, 990\},$$
$$\text{and } F = \{13, 26, \ldots, 988\}.$$

Find the number of elements in the set $(((((A \cup B) \cap C) \cup D) \cap E) \cup F)$.

I8) Lason Jiu gives a problem to Sick Nong and Ayush Agrawal. Sick takes 6 minutes to solve the problem, while Ayush takes 9 minutes. Sick has a 1/3 chance of solving correctly and Ayush has a 2/3 chance of solving correctly. If they solved it incorrectly, they resume solving with the same time and accuracy. Lason gives a rubber chicken to the first person who solves it correctly. If Sick and Ayush solve the question at the same time, Lason checks Sick's work first. The probability that Ayush wins the rubber chicken can be expressed as $\frac{p}{q}$. Find $p+q$.

I9) Find the maximum value of x such that x divides all $p^{32}-1$ for all primes $p > 20$.

I10) Let $\overline{AB}$ be a line segment of length 2, C_1 be the circle with diameter $\overline{AB}$, C_0 be the circle with radius 2 externally tangent to C_1 at A, and C_2 be the circle with radius 3 externally tangent to C_1 at B. Duck D_1 is located at point B, Duck D_2 is located on C_2, 270 degrees counterclockwise from B, and Duck D_0 is located on C_0, 270 degrees counterclockwise from A. At the same time, the ducks all start running counterclockwise around their corresponding circles, with each duck taking the same amount of time to complete a full lap around its circle. When the 3 ducks are first collinear, the distance between D_0 and D_2 can be expressed as $p\sqrt{q}$. Find $p+q$.



I11) A river is bounded by the lines $x = 0$ and $x = 25$, with a current of 2 units/s in the positive y-direction. At $t = 0$, a mallard is at $(0, 0)$, and a wigeon is at $(25, 0)$. They start swimming with a constant speed such that they meet at $(x, 22)$. The mallard has a speed of 4 units/s relative to the water, and the wigeon has a speed of 3 units/s relative to the water. Find the value of x.

I12) Find the number of 8-tuples of binary inputs $\{A, B, C, D, E, F, G, H\}$ such that

$$\{(A \text{ AND } B) \text{ OR } (C \text{ AND } D)\} \text{ AND } \{(E \text{ AND } F) \text{ OR } (G \text{ AND } H)\}$$

$$= \{(A \text{ OR } B) \text{ AND } (C \text{ OR } D)\} \text{ OR } \{(E \text{ OR } F) \text{ AND } (G \text{ OR } H)\}$$

The AND gates produce an output that is ON only if both the inputs are ON, and the OR gates produce an output that is OFF only if both inputs are OFF.

I13) Let triangle A_1BC have sides $A_1B = 5$, $A_1C = 12$, and $BC = 13$. For all natural numbers i, let B_i be the foot of the altitude from A_i to BC, let A_{2i} be the foot of the altitude from B_i to A_1B, and let A_{2i+1} be the foot of the altitude from B_i to A_1C.

$$\sum_{i=1}^{7} A_iB_i = \frac{p}{q}$$

Find $p + q$.

I14) Let ABC be a triangle with side lengths $AB = 6, AC = 7$, and $BC = 8$. Let H be the orthocenter of $\triangle ABC$ and H' be the reflection of H across the midpoint M of BC. $\frac{[ABH']}{[ACH']}$ can be expressed as $\frac{p}{q}$. Find $p + q$.

I15) Let P, A, B, C, D be points on a plane such that $PA = 9$, $PB = 19$, $PC = 9$, $PD = 5$, $\angle APB = 120°$, $\angle BPC = 45°$, $\angle CPD = 60°$, and $\angle DPA = 135°$. Let G_1, G_2, G_3, and G_4 be the centroids of triangles PAB, PBC, PCD, PDA. $[G_1G_2G_3G_4]$ can be expressed as $a\sqrt{b} + c\sqrt{d}$. Find $a + b + c + d$.