Team Round

CCA Math Bonanza

23 Apr 2022

T1) Let a, b, c, and d be positive integers such that $77^a \cdot 637^b = 143^c \cdot 49^d$. Compute the minimal value of $a + b + c + d$.

T2) CCA's B building has 6 rooms on the second floor, labeled B201 to B206, as well as 8 rooms on the first floor, labeled B101 to B108. Annie is currently in room B205. Each minute, she chooses to stay or change floors with equal probability, and chooses a classroom on that floor to go to at random (she can stay in the classroom that she's already in). B104, B108, and B203 are the only rooms that have teachers who will scold her for randomly walking around during class time. The probability that she is first scolded in room B203 can be expressed as $\frac{p}{q}$. Compute $p + q$.

T3) The smallest possible volume of a cylinder that will fit nine spheres of radius 1 can be expressed as $x\pi$ for some value of x. Compute x.

T4) Let there exist a configuration of **exactly** 1 black king, n black chess pieces (each of which can be a pawn, knight, bishop, rook, or queen), and a white *anti-king* on a standard 8x8 board in which the white *anti-king* is not under attack, but will be if it is moved. Compute the minimal value of n.

*An *anti-king* can move to any square is **not** 1 square vertically, horizontally, or diagonally. It can also capture undefended pieces.

T5) Maggie Waggie organizes a pile of 127 calculus tests in alphabetical order, with Joccy Woccy's test being 64th in the pile. While Maggie isn't looking, Joccy walks over and randomly scrambles the entire pile of tests. When Maggie returns, she is oblivious to the fact that Joccy has tampered with the list. She uses a binary search algorithm to find Joccy's test, where she looks at the test in the middle of the pile. If the test is not Joccy's, she binary searches the top half of the list if the test appears after Joccy's name when arranged alphabetically, or the bottom half of the list otherwise. The probability that Maggie finds Joccy's test can be expressed as $\frac{p}{q}$. Compute $p + q$.

T6) A bird starts with 300 ml of blood at 100 degrees in its body, 50 ml of blood at 0 degrees in its feet. Every minute, 50 ml of blood flows from the body to the feet, and 50 ml of blood at 40% of the body temperature flows from the feet to the body. The bird feels cold once its internal body temperature (**not** including the feet) falls below 60 degrees. Compute how many minutes it takes for the bird to feel cold.

T7) A caretaker is giving candy to his two babies. Every minute, he gives a candy to one of his two babies at random. The five possible moods for the babies from to be in, from saddest to happiest, are "upset," "sad," "okay," "happy," and "delighted." A baby gets happier by one mood when they get a candy and gets sadder by one mood when the other baby gets one. Both babies start at the "okay" state, and a baby will start crying if they don't get a candy when they're already "upset". The probability that 10 minutes pass without either baby crying can be expressed as $\frac{p}{q}$. Compute $p+q$.

T8) Let n be a set of integers. $S(n)$ is defined as the sum of the elements of n. $T = \{1,2,3,4,5,6,7,8,9\}$ and A and B are subsets of T such that A $\cup B = T$ and A $\cap B = \varnothing$. The probability that $S(A) \geq 4S(B)$ can be expressed as $\frac{p}{q}$. Compute $p+q$.

T9) Equilateral octagon $A_1A_2A_3A_4A_5A_6A_7A_8$ is constructed such that $A_1A_3A_5A_7$ is a square of side length $\sqrt{2}$ and $A_2A_4A_6A_8$ is a square of side length $4/3$. For each vertex A_i of the octagon, let B_i be the intersection of lines $A_{i+1}A_{i+2}$ and $A_{i-1}A_{i-2}$, where $A_{i-8} = A_i = A_{i+8}$. Compute $[B_1B_2B_3B_4B_5B_6B_7B_8]^2$.

T10) Evan, Larry, and Alex are drawing whales on the whiteboard. Evan draws 10 whales, Larry draws 15 whales, and Alex draws 20 whales. Michelle then starts randomly erasing whales one by one. The probability that she finishes erasing Larry's whales first can be expressed as $\frac{p}{q}$. Compute $p+q$.